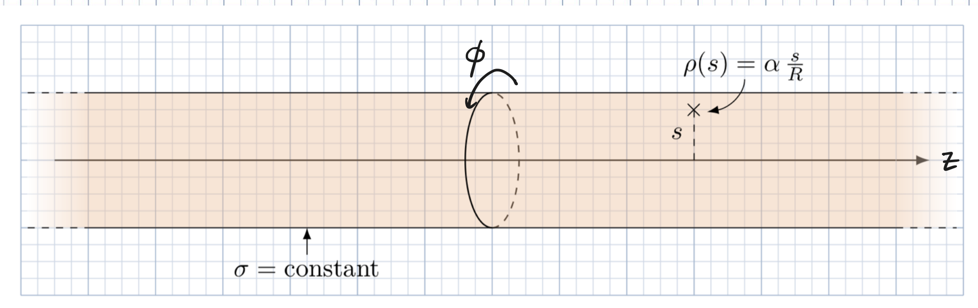
Post-Exam Practice Problems

1 Use Gauss's Law to find $\vec{E}$ inside & outside this very ($\sim\infty$) long charged cylinder.

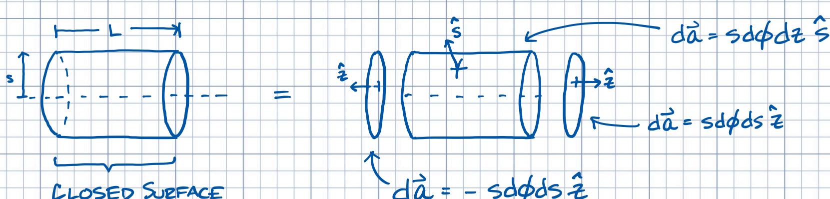


- Cylindrical symmetry tells us that $\vec{E}$ can only depend on s, but not ϕ or z. Likewise, it can only point in the $\hat{s}$ direction. (What would tell it to point in the $+\hat{z}$ rather than the $-\hat{z}$ direction?) So whether we're @ a point inside ($s<R$) or outside ($s>R$) the cylinder, $\vec{E}$ will have the form:

$$\vec{E} = E(s)\,\hat{s}$$

- We'll use Gauss's Law to find $E(s)$, which might behave differently for $s<R$ & $s>R$.

- Our Gaussian surfaces should be cylinders w/ the <u>same</u> axis as above, radius s, and length L:



- The discs on the end of the cylinder don't contribute to the flux since $\pm\hat{z}\cdot\hat{s} = 0$. All the flux is through the wall of the cylinder:

$$\Phi_E = \oint_{G.S.} d\vec{a}\cdot\vec{E} = \int_{\substack{\text{Left}\\ \text{End}}} da\,\cancel{(-\hat{z})\cdot(E(s)\hat{s})}^{\,0} + \int_{\substack{\text{Right}\\ \text{End}}} da\,\cancel{\hat{z}\cdot(E(s)\hat{s})}^{\,0}$$

$$+ \int_0^L dz \int_0^{2\pi} d\phi\; s\; \hat{s}\cdot(E(s)\hat{s})$$

$$\Phi_E = 2\pi s L\, E(s)$$

We don't yet know E(s), but we know how it is related to the flux through this G.S.

- Gauss's Law tells us that the flux of $\vec{E}$ through any closed surface is proportional to the amount of charge inside the surface. So how much charge is in our Gaussian surface? It depends on s!

- If $s < R$, then our G.S. encloses some of the ρ inside the charged cylinder:

$$q_{enc}(s<R) = \int_{G.V.} d\tau'\rho(\vec{r}\,') = \int_0^{2\pi} d\phi' \int_0^L dz' \int_0^s ds'\, \underbrace{s'}_{\text{From } d\tau'}\; \underbrace{\rho_0 \frac{s'}{R}}_{\rho(\vec{r}\,')}$$

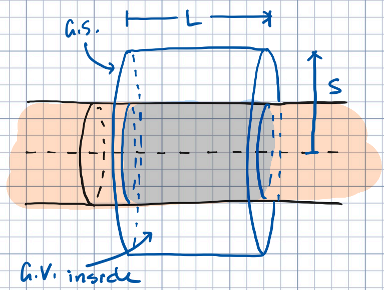
$$= 2\pi L \rho_0 \frac{1}{R}\left(\frac{1}{3} s'^3 \Big|_0^s\right) = \frac{2}{3}\pi L \rho_0 \frac{s^3}{R}$$

GAUSS: $$2\pi L s\, E(s) = \frac{1}{\varepsilon_0} q_{enc} = \frac{2\pi L \rho_0 s^3}{3\varepsilon_0 R}$$

$$\hookrightarrow \boxed{\vec{E}(s<R) = \frac{\rho_0}{3\varepsilon_0}\frac{s^2}{R}\hat{s}}$$

NOTICE THAT L CANCELS! It's just a length I chose for my G.S. I could have used any value – 1m, 1cm, etc – $\vec{E}$ can't depend on it!

- If $s > R$, the G.S. encloses all the volume charge out to R ($\rho = 0$ for $s > R$) as well as the charge on the surface of the charged cylinder:



$$q_{enc}(s>R) = \int_{\substack{\text{inside}\\ \text{G.V.}}} dq(\vec{r}\,')$$

$$= \int_0^{2\pi} d\phi' \int_0^L dz' \int_0^R ds'\, s'\, \rho(s') + \int_0^{2\pi} d\phi' \int_0^L dz'\, R\, \sigma$$

Constant (otherwise, would not have cylindrical symmetry!)

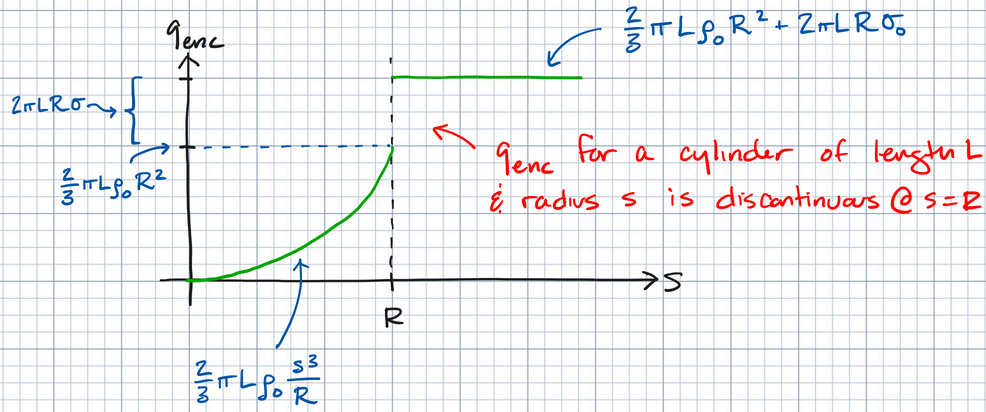
$$= \underbrace{\frac{2}{3}\pi L \rho_0 R^2}_{\text{From } \rho = \rho_0 \frac{s}{R} \text{ in cylinder w/ } 0 \le s \le R} + \underbrace{2\pi L R \sigma}_{\text{Const. } \sigma \text{ on surface @ } s=R}$$

GAUSS: $\cancel{2}\cancel{\pi}\cancel{L}\, s\, E(s) = \dfrac{\cancel{2}\cancel{\pi}\cancel{L}\rho_0 R^2}{3\varepsilon_0} + \dfrac{\cancel{2}\cancel{\pi}\cancel{L} R\sigma}{\varepsilon_0}$

$$\hookrightarrow \boxed{\vec{E}(s>R) = \left(\frac{\rho_0 R^2}{3\varepsilon_0} + \frac{\sigma R}{\varepsilon_0}\right)\frac{1}{s}\,\hat{s}}$$

- These results make sense. Inside, $s < R$, moving away from the axis means there is more charge contributing to $\vec{E}$ (enclosed by the GS) so it grows. But once $s > R$, a larger GS doesn't enclose any more charge. The area of the GS grows, but the enclosed charge & hence the flux don't change, so $\vec{E}$ has to decrease like $1/s$ so that $\Phi_E = 2\pi L s E(s)$ stays the same.

- One other thing happens as soon as $s > R$. The q_{enc} suddenly 'jumps' b/c of the constant σ on the surface $s = R$.



- Since q_{enc} suddenly jumps by a finite amount ($2\pi L R\sigma$, which could be pos. or neg. depending on the sign of σ), Gauss's Law tells us that the FLUX through our G.S. must also see a jump as soon as $s > R$. And since Φ is proportional to $E(s)$, that means $\vec{E}$ also has a discontinuity. We see this in our expressions for $\vec{E}$!

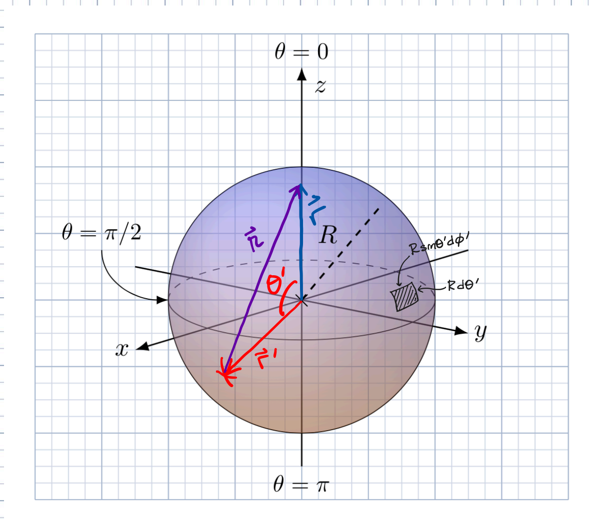
$$\vec{E}_{out}(s=R) = \left(\frac{\rho_o R}{3\varepsilon_o} + \frac{\sigma}{\varepsilon_o}\right)\hat{s}$$

$$\vec{E}_{in}(s=R) = \frac{\rho_o R}{3\varepsilon_o}\hat{s}$$

$$\Rightarrow \vec{E}_{out}(s=R) - \vec{E}_{in}(s=R) = \frac{\sigma}{\varepsilon_o}\hat{s} \quad \checkmark$$

- As long as we treat the surface charge like an ∞-thin layer, the $\vec{E}$ is discontinuous @ $s = R$.

$\boxed{2}$ Sphere w/ charge density $\sigma(\theta) = \sigma_0 \cos\theta$.



A) TOTAL CHARGE?

$$dq(\theta',\phi') = \underbrace{da'}_{R^2 \sin\theta' d\theta' d\phi'} \sigma_0 \cos\theta'$$

$$q_{TOT} = \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' R^2 \sigma_0 \sin\theta' \cos\theta'$$

$$= 0$$

B) V @ $\vec{r} = z\hat{z}$, $z > 0$?

$$\vec{r} = z\hat{z} \qquad \vec{r}\,' = R\,\hat{r}(\theta',\phi')$$

$$\mathscr{r} = \sqrt{(\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')} = \sqrt{\vec{r}\cdot\vec{r} + \vec{r}\,'\cdot\vec{r}\,' - 2\vec{r}\cdot\vec{r}\,'} = \sqrt{z^2 + R^2 - 2Rz\cos\theta'}$$

$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int dq(\vec{r}\,')\frac{1}{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' R^2 \sin\theta' \frac{\sigma_0 \cos\theta'}{\sqrt{z^2+R^2-2Rz\cos\theta'}}$$

$$= \frac{2\pi}{4\pi\varepsilon_0} R^2 \sigma_0 \int_0^{\pi} d\theta' \sin\theta' \frac{\cos\theta'}{\sqrt{z^2+R^2-2Rz\cos\theta'}}$$

$$u = z^2 + R^2 - 2Rz\cos\theta' \qquad du = 2Rz\sin\theta' d\theta'$$

$$\theta' = 0 \rightarrow u = z^2 + R^2 - 2Rz = (z-R)^2, \quad \theta' = \pi \rightarrow u = (z+R)^2$$

$$\rightarrow V(0,0,z) = \frac{R^2\sigma_0}{2\varepsilon_0}\int_{(z-R)^2}^{(z+R)^2} du \frac{1}{2Rz}\frac{1}{\sqrt{u}}\Big(\underbrace{\frac{z^2+R^2-u}{2Rz}}_{\cos\theta'}\Big)$$

$$V(0,0,z) = \frac{R^2\sigma_0}{2\varepsilon_0}\int_{(z-R)^2}^{(z+R)^2} du\, \frac{1}{2Rz}\frac{1}{\sqrt{u}}\left(\frac{z^2+R^2-u}{2Rz}\right)$$

$$= \frac{R\sigma_0}{4z\varepsilon_0}\int_{(z-R)^2}^{(z+R)^2} du\left[\left(\frac{z^2+R^2}{2Rz}\right)\frac{1}{\sqrt{u}} - \frac{1}{2Rz}\sqrt{u}\right]$$

$$= \frac{R\sigma_0}{4z\varepsilon_0}\left[\left(\frac{z^2+R^2}{\cancel{2}Rz}\right)\cancel{2}\sqrt{u}\,\Big|_{(z-R)^2}^{(z+R)^2} - \frac{1}{\cancel{2}Rz}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_{(z-R)^2}^{(z+R)^2}\right]$$

Now we have to be careful: R is positive, and the problem says $z>0$, so $z+R$ is definitely a positive number. We can safely say that:

$$\sqrt{(z+R)^2} = z+R \qquad \left((z+R)^2\right)^{3/2} = (z+R)^3 = z^3+3z^2R+3zR^2+R^3$$

<u>But</u>, $z-R$ could be positive if $z>R$, or negative if $z<R$. So

$$\sqrt{(z-R)^2} = \begin{cases} z-R, & z>R \\ R-z, & z<R \end{cases} \qquad \left((z-R)^2\right)^{3/2} = \begin{cases} z^3-3z^2R+3zR^2-R^3, & z>R \\ R^3-3zR^2+3z^2R-z^3, & z<R \end{cases}$$

So:

$$V(0,0,z\geq R) = \frac{R\sigma_0}{4\varepsilon_0 z}\left[\frac{(z^2+R^2)}{Rz}\Big(\overbrace{z+R-(z-R)}^{2R}\Big) - \frac{1}{3Rz}\Big(\underbrace{(z+R)^3-(z-R)^3}_{2R^3+6Rz^2}\Big)\right]$$

$$= \frac{R\sigma_0}{4\varepsilon_0 z}\times\left[2\frac{(\cancel{z^2}+R^2)}{z} - \frac{2}{3}\frac{R^2}{z} - \cancel{2z}\right]$$

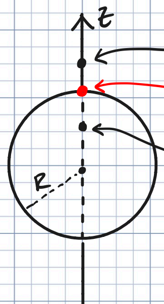
$$\Rightarrow V(0,0,z\geq R) = \frac{\sigma_0 R^3}{3\varepsilon_0 z^2}$$

$$V(0,0,z\leq R) = \frac{R\sigma_0}{4\varepsilon_0 z}\left[\frac{(z^2+R^2)}{Rz}\Big(\overbrace{z+R-(R-z)}^{2z}\Big) - \frac{1}{3Rz}\Big(\underbrace{(z+R)^3-(R-z)^3}_{2z^3+6zR^2}\Big)\right]$$

$$= \frac{R\sigma_0}{4\varepsilon_0 z}\times\left[2\,\frac{(z^2+\cancel{R^2})}{R} - \frac{2}{3}\,\frac{z^2}{R} - \cancel{2R}\right]$$

$$\Rightarrow \boxed{V(0,0,z\leq R) = \frac{\sigma_0 z}{3\varepsilon_0}}$$

Notice that these expressions agree @ $z=R$. They have to – the potential is always continuous.



$$V(0,0,z\geq R) = \frac{\sigma_0 R^3}{3\varepsilon_0 z^2}$$

Both $= \frac{\sigma_0 R}{3\varepsilon_0}$ @ $z=R$

$$V(0,0,z\leq R) = \frac{\sigma_0 z}{3\varepsilon_0}$$

We assumed $z>0$ here to keep things simple. However, we recently solved this problem using S.O.V. & found:

$$V(r\leq R) = \frac{\sigma_0}{3\varepsilon_0}\, r\cos\theta$$

$$V(r\geq R) = \frac{\sigma_0}{3\varepsilon_0}\,\frac{R^3}{r^2}\cos\theta$$

c) To find $\vec{E}(0,0,z)$ we take $-\vec{\nabla}V$. We don't expect to get an E_x or E_y @ $\vec{r} = z\hat{z}$ (symmetry) so it should be okay that we're using $V(0,0,z)$ rather than $V(x,y,z)$!

$$-\vec{\nabla}V(0,0,z\leq R) = -\hat{z}\frac{d}{dz}\left(\frac{\sigma_0 z}{3\varepsilon_0}\right) = -\frac{\sigma_0}{3\varepsilon_0}\hat{z}$$

$$-\vec{\nabla}V(0,0,z\geq R) = -\hat{z}\frac{d}{dz}\left(\frac{\sigma_0}{3\varepsilon_0}\frac{R^3}{z^2}\right) = \frac{2}{3}\frac{\sigma_0}{\varepsilon_0}\frac{R^3}{z^3}\hat{z}$$

$$\hookrightarrow \vec{E}(0,0,z) = \begin{cases} -\frac{\sigma_0}{3\varepsilon_0}\hat{z}, & 0 \leq z < R \\ \frac{2}{3}\frac{\sigma_0}{\varepsilon_0}\frac{R^3}{z^3}\hat{z}, & R < z \end{cases}$$

These disagree @ $z=R$ b/c of the surface charge:

$$\vec{E}_{out}(0,0,R) - \vec{E}_{in}(0,0,R) = \frac{2}{3}\frac{\sigma_0}{\varepsilon_0}\hat{z} - \left(-\frac{1}{3}\frac{\sigma_0}{\varepsilon_0}\hat{z}\right)$$

$$\checkmark = \frac{\sigma_0}{\varepsilon_0}\hat{z}$$

σ @ $(0,0,R)$

$\hat{r}$ @ $(0,0,R)$